

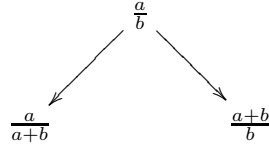
# FORESTS OF COMPLEX NUMBERS

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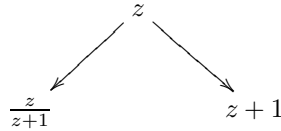
**ABSTRACT.** The Calkin-Wilf tree is an infinite binary tree whose vertices are the positive rational numbers. Each number occurs in the tree exactly once and in the form  $a/b$ , where  $a$  and  $b$  are relatively prime positive integers. In this paper, certain subsemigroups of the modular group are used to construct similar trees in the set  $\mathcal{D}_0$  of positive complex numbers. Associated to each semigroup is a forest of trees that partitions  $\mathcal{D}_0$ . The fundamental domain and the set of cusps of the semigroup are defined and computed.

## 1. FORESTS GENERATED BY LEFT-RIGHT PAIRS OF MATRICES

Let  $\mathbf{N}$ ,  $\mathbf{N}_0$ ,  $\mathbf{Z}$ , and  $\mathbf{Q}$  denote, as usual, the sets of positive integers, nonnegative integers, integers, and rational numbers, respectively. The Calkin-Wilf tree is a rooted infinite binary tree whose vertices are the positive rational numbers. The root of the tree is 1, and the generation rule is



Writing  $z = a/b$ , we can redraw this as follows:



Every positive rational number occurs exactly once as a vertex in the Calkin-Wilf tree, and the geometry of the tree encodes beautiful arithmetical relations between rational numbers (Bates, Bunder, and Tognetti [1], Bates and Mansour [2], Calkin and Wilf [4], Chan [5], Dilcher and Stolarsky [6], Gibbons, Lester, and Bird [7], Han, Masuda, Singh, and Thiel [8], Mallows [10], Mansour and Shattuck [11], Nathanson [12, 14, 13], Reznick [15]).

Let

$$SL_2(\mathbf{N}_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{N}_0 \text{ and } ad - bc = 1 \right\}$$

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be the semigroup of  $2 \times 2$  matrices with nonnegative integral coordinates and determinant 1. Every matrix  $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{N}_0)$  defines a function  $z \mapsto T(z)$  by

$$T(z) = \frac{az + b}{cz + d}.$$

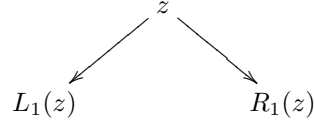
For example, if

$$L_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then

$$L_1(z) = \frac{z}{z+1} \quad \text{and} \quad R_1(z) = z+1.$$

Thus, the generation rule for the Calkin-Wilf tree can also be presented in the form



Denote the real and imaginary parts of a complex number  $z$  by  $\Re(z)$  and  $\Im(z)$ , respectively. Let  $K$  be a non-real subfield of the complex numbers, that is,  $K \subseteq \mathbf{C}$  and  $K \not\subseteq \mathbf{R}$ . In this paper we consider the set of “positive complex numbers”

$$\mathcal{D}_0 = \{z = x + yi \in K : x > 0 \text{ and } y > 0\}.$$

**Theorem 1.** *If  $z \in \mathcal{D}_0$  and  $T \in SL_2(\mathbf{N}_0)$ , then  $T(z) \in \mathcal{D}_0$ . Moreover, the function  $(T, z) \mapsto T(z)$  from  $SL_2(\mathbf{N}_0) \times \mathcal{D}_0$  into  $\mathcal{D}_0$  defines a semigroup action of  $SL_2(\mathbf{N}_0)$  on the set  $\mathcal{D}_0$ .*

*Proof.* If  $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{N}_0)$  and if  $z = x + yi \in \mathcal{D}_0$ , then  $x > 0$ ,  $y > 0$ , and  $ad - bc = 1$ . A standard calculation gives

$$\begin{aligned} T(z) &= \frac{az + b}{cz + d} = \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2} \\ &= \frac{ac|z|^2 + (ad + bc)x + bd}{|cz + d|^2} + \frac{yi}{|cz + d|^2}. \end{aligned}$$

Because  $ac|z|^2 + (ad + bc)x + bd$ ,  $|cz + d|^2$ , and  $y$  are positive real numbers, it follows that  $T(z) \in \mathcal{D}_0$ . Because  $(T_1 T_2)z = T_1(T_2 z)$  for all  $T_1, T_2 \in SL_2(\mathbf{N}_0)$  and  $z \in \mathcal{D}_0$ , the function  $(T, z) \mapsto T(z)$  from  $SL_2(\mathbf{N}_0) \times \mathcal{D}_0$  into  $\mathcal{D}_0$  defines a semigroup action of  $SL_2(\mathbf{N}_0)$  on the set  $\mathcal{D}_0$ . This completes the proof.  $\square$

**Lemma 1.** *If  $T \in SL_2(\mathbf{N}_0)$  and  $T \neq I$ , then the fractional linear transformation  $z \mapsto T(z)$  has no fixed points in  $\mathcal{D}_0$ .*

*Proof.* Let  $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , where  $a, b, c, d \in \mathbf{N}_0$  and  $ad - bc = 1$ . If  $z$  is a fixed point of  $T$ , then

$$\frac{az + b}{cz + d} = z.$$

Equivalently,

$$cz^2 + (d - a)z - b = 0.$$

If  $c = 0$ , then  $a = d = 1$  and so  $z = T(z) = z + b$ , hence  $b = 0$  and  $T = I$ , which is absurd. If  $c \neq 0$ , then

$$\begin{aligned} z &= \frac{a - d \pm \sqrt{(d - a)^2 + 4bc}}{2c} \\ &= \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}. \end{aligned}$$

If  $z \in \mathcal{D}_0$ , then  $\Im(z) > 0$  and so  $(a + d)^2 - 4 < 0$ . Equivalently,  $a + d = |a + d| < 2$ . Because  $a, b, c$ , and  $d$  are nonnegative integers, it follows that  $0 = ad = 1 + bc \geq 1$ , which is also absurd. Therefore,  $T$  has no fixed points in  $\mathcal{D}_0$ .  $\square$

Let  $L$  and  $R$  be non-identity matrices in  $SL_2(\mathbf{N}_0)$ . The ordered pair  $(L, R)$  will be called a *left-right pair* if

$$L(\mathcal{D}_0) \cap R(\mathcal{D}_0) = \emptyset.$$

Equivalently,  $(L, R)$  is a *left-right pair* if

$$L(z_1) \neq R(z_2) \quad \text{for all } z_1, z_2 \in \mathcal{D}_0.$$

**Lemma 2.** *If  $(L, R)$  is a left-right pair, then the subsemigroup  $\langle L, R \rangle$  of  $SL_2(\mathbf{N}_0)$  generated by  $\{L, R\}$  is free.*

*Proof.* The semigroup  $\langle L, R \rangle$  is free if and only if the unique solution of the matrix equation

$$(1) \quad T_1 T_2 \cdots T_k = T'_1 T'_2 \cdots T'_\ell$$

with  $k, \ell \in \mathbf{N}_0$  and  $T_i, T'_j \in \{L, R\}$  for all  $i \in \{1, 2, \dots, k\}$  and  $j \in \{1, 2, \dots, \ell\}$  is the trivial solution  $k = \ell$  and  $T_i = T'_i$  for all  $i \in \{1, 2, \dots, k\}$ . If there is a nontrivial solution, then there is a minimal solution, that is, a matrix identity (1) with  $k \geq \ell$  and  $k$  minimal.

If  $\ell \geq 1$ , then for every  $z \in \mathcal{D}_0$  we have

$$T_1 T_2 \cdots T_k(z) = T'_1 T'_2 \cdots T'_\ell(z)$$

and so

$$T_1(z_1) = T'_1(z_2)$$

where

$$z_1 = T_2 \cdots T_k(z) \in \mathcal{D}_0 \quad \text{and} \quad z_2 = T'_2 \cdots T'_\ell(z) \in \mathcal{D}_0.$$

Because  $(L, R)$  is a left-right pair, it follows that  $T_1 = T'_1$ , and so  $T_2 \cdots T_k = T'_2 \cdots T'_\ell$ , which contradicts the minimality of  $k$ .

If  $\ell = 0$ , then  $k \geq 1$  and  $T_1 T_2 \cdots T_k = I$ . Let  $T_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{N}_0)$ . Then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = T_1^{-1} = T_2 \cdots T_k \in SL_2(\mathbf{N}_0)$$

and so  $b = c = 0$  and  $a = d = 1$ , that is,  $T_1 = I \in \{L, R\}$ , which is absurd. This completes the proof.  $\square$

**Lemma 3.** *For positive integers  $u$  and  $v$ , let*

$$(2) \quad L_u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \quad \text{and} \quad R_v = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

*Then  $(L_u, R_v)$  is a left-right pair.*

*Proof.* If  $z = x + yi \in \mathcal{D}_0$ , then

$$L_u(z) = \frac{z}{uz + 1} = \frac{u(x^2 + y^2) + x}{|uz + 1|^2} + \frac{yi}{|uz + 1|^2}.$$

Because

$$u(x^2 + y^2) + x < u^2(x^2 + y^2) + 2ux + 1 = (ux + 1)^2 + (uy)^2 = |uz + 1|^2$$

it follows that

$$\Re(L_u(z)) < 1.$$

Similarly,  $R_v(z) = (x + v) + yi$  and so

$$\Re(R_v(z)) = x + v > v \geq 1.$$

Therefore,

$$\Re(L_u(z_1)) < 1 < \Re(R_v(z_2))$$

for all  $z_1, z_2 \in \mathcal{D}_0$ , and so  $L(\mathcal{D}_0) \cap R(\mathcal{D}_0) = \emptyset$ . This completes the proof.  $\square$

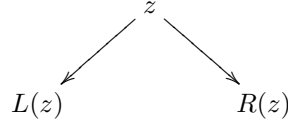
Lemmas 2 and 3 imply that the set  $\{L_u, R_v\}$  freely generates the semigroup  $\langle L_u, R_v \rangle$ . Note that  $L_u = L_1^u$  and  $R_v = R_1^v$ . It is a classical result that  $\{L_1, R_1\}$  freely generates  $SL_2(\mathbf{N}_0)$ , and this also proves that  $\{L_u, R_v\}$  freely generates  $\langle L_u, R_v \rangle$  (cf. Nathanson [14]).

Let  $(L, R)$  be a pair of matrices in  $SL_2(\mathbf{N}_0)$ . We consider the directed graph  $\mathcal{F}(L, R)$  whose vertex set is  $\mathcal{D}_0$  and whose edge set is

$$\{(z, L(z)) : z \in \mathcal{D}_0\} \cup \{(z, R(z)) : z \in \mathcal{D}_0\}.$$

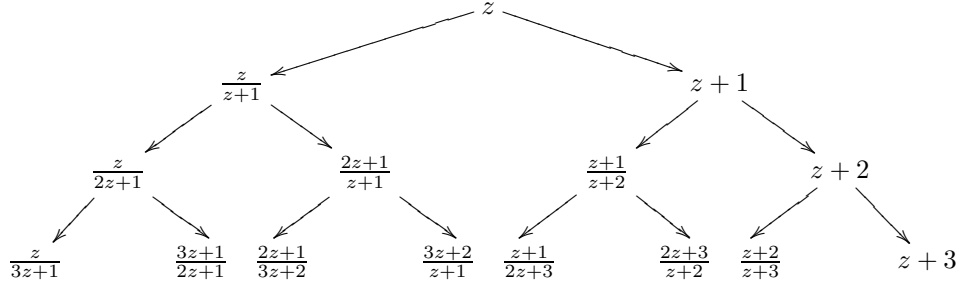
In this graph, every vertex has outdegree 2:

(3)



We call  $L(z)$  the *left child* of  $z$  and  $R(z)$  the *right child* of  $z$ , and we call  $z$  the *parent* of  $L(z)$  and of  $R(z)$ . Because  $L$  and  $R$  are invertible matrices, if  $L(z_1) = L(z_2)$  or if  $R(z_1) = R(z_2)$  for some  $z_1, z_2 \in \mathcal{D}_0$ , then  $z_1 = z_2$ . If  $(L, R)$  is a right-left pair, then there do not exist  $z_1, z_2 \in \mathcal{D}_0$  such that  $L(z_1) = R(z_2) = z$ , and so the indegree of  $z$  is either 0 or 1. We call  $z$  an *orphan* if it has no parent, that is, if  $z$  has indegree 0.

For example, let  $(L_1, R_1)$  be the pair of matrices defined by (2) with  $u = v = 1$ . The first three generations of descendants of the complex number  $z \in \mathcal{D}_0$  are:



If  $z$  is a variable, then the vertices of this tree are the linear fractional transformations associated with the semigroup  $SL_2(\mathbf{N}_0)$ . Nathanson [12] described some remarkable arithmetical properties of this tree.

**Theorem 2.** *For every left-right pair  $(L, R)$  of matrices in  $SL_2(\mathbf{N}_0)$ , the directed graph  $\mathcal{F}(L, R)$  is a forest of infinite binary trees.*

*Proof.* We must prove that every connected component of the graph is a tree. If not, then some component of the graph contains an undirected cycle of length  $n \geq 2$ , that is, a sequence of  $n \geq 2$  vertices  $z_0, z_1, \dots, z_{n-1}, z_n$  such that

- (i)  $z_i \neq z_j$  for  $0 \leq i < j \leq n-1$  and  $z_n = z_0$ ;
- (ii) for  $i = 0, 1, \dots, n-1$ , either  $(z_i, z_{i+1})$  is an edge or  $(z_{i+1}, z_i)$  is an edge.

Suppose that  $(z_0, z_1)$  is an edge. Then  $z_0$  is the parent of  $z_1$ . Because  $(L, R)$  is a left-right pair, every vertex has at most one parent. This implies that  $z_1$  is the parent of  $z_2$ , and so  $z_2$  is the parent of  $z_3$ . Continuing inductively, we conclude that  $z_i$  is the parent of  $z_{i+1}$  for  $i = 0, 1, \dots, n-1$ . Thus,  $(z_0, z_1, \dots, z_n)$  is not only a cycle in the graph, but is a directed cycle. It follows that there is a sequence of matrices  $T_0, T_1, \dots, T_{n-1}$  such that  $T_i \in \{L, R\}$  and  $T_i z_i = z_{i+1}$  for all  $i = 0, 1, \dots, n-1$ . Thus,  $T = T_{n-1} \cdots T_1 T_0 \in SL_2(\mathbf{N}_0)$  and  $T(z_0) = z_0$ , that is,  $z_0 \in \mathcal{D}_0$  is a fixed point of  $T$ . Lemma 1 implies that  $T = I$  and so

$$T_0^{-1} = T_{n-1} \cdots T_1 \in SL_2(\mathbf{N}_0)$$

which is impossible because (as observed in the proof of Lemma 2) the only invertible matrix in  $SL_2(\mathbf{N}_0)$  is the identity matrix  $I$ . The same argument applies if  $(z_1, z_0)$  is an edge. Thus, every component of the directed graph  $\mathcal{F}(L, R)$  is a tree, and so  $\mathcal{F}(L, R)$  is a forest. Because every vertex in  $\mathcal{F}(L, R)$  has outdegree 2, it follows that every component of  $\mathcal{F}(L, R)$  is an infinite binary tree.  $\square$

Let  $(L, R)$  be a left-right pair, and let  $\langle L, R \rangle$  be the subsemigroup of  $SL_2(\mathbf{N}_0)$  generated by  $\{L, R\}$ . Every complex number  $w \in \mathcal{D}_0$  is the root of an infinite binary tree whose vertices are the complex numbers in  $\mathcal{D}_0$  constructed from the generation rule (3). The *orbit* of  $w$ , denoted  $\text{orbit}(w)$ , is the set of vertices in this tree. Equivalently,

$$\text{orbit}(w) = \{T(w) : T \in \langle L, R \rangle\}.$$

**Lemma 4.** *Let  $(L, R)$  be a left-right pair, and consider the forest  $\mathcal{F}(L, R)$ . Let  $w_1, w_2 \in \mathcal{D}_0$ . If  $\text{orbit}(w_1) \cap \text{orbit}(w_2) \neq \emptyset$ , then either  $\text{orbit}(w_1) \subseteq \text{orbit}(w_2)$  or  $\text{orbit}(w_2) \subseteq \text{orbit}(w_1)$ .*

*Proof.* If  $\text{orbit}(w_1) \cap \text{orbit}(w_2) \neq \emptyset$ , then there exist matrices  $T, T' \in \langle L, R \rangle$  such that

$$T(w_1) = T'(w_2).$$

There are sequences of matrices  $(T_i)_{i=1}^k$  and  $(T'_j)_{j=1}^\ell$  such that  $T_i, T'_j \in \{L, R\}$  for  $i = 1, \dots, k$  and  $j = 1, \dots, \ell$ , with  $T = T_1 T_2 \cdots T_k$  and  $T' = T'_1 T'_2 \cdots T'_\ell$ . Therefore,

$$T_1 (T_2 \cdots T_k (w_1)) = T(w_1) = T'(w_2) = T'_1 (T'_2 \cdots T'_\ell (w_2))$$

Because  $(L, R)$  is a left-right pair, it follows that  $T_1 = T'_1$  and so  $T_2 \cdots T_k (w_1) = T'_2 \cdots T'_\ell (w_2)$ . Suppose that  $k \geq \ell$ . Continuing inductively, we obtain  $T_{\ell+1} \cdots T_k (w_1) = w_2$ , and so  $w_2 \in \text{orbit}(w_1)$ . It follows that  $\text{orbit}(w_2) \subseteq \text{orbit}(w_1)$ . This completes the proof.  $\square$

Let  $(L, R)$  be a left-right pair, and let  $\mathcal{F}(L, R)$  be the associated forest whose vertices are the complex numbers in  $\mathcal{D}_0$ . Let  $w \in \mathcal{D}_0$ . Every element in  $\text{orbit}(w) \setminus \{w\}$  is a *descendant* of  $w$ , and  $w$  is an *ancestor* of every element in  $\text{orbit}(w) \setminus \{w\}$ . An *orphan* is a complex number in  $\mathcal{D}_0$  with no ancestors. A complex number in  $\mathcal{D}_0$  is a descendant of an orphan if and only if it has only finitely many ancestors. There is a one-to-one correspondence between the rooted infinite binary trees in the forest  $\mathcal{F}(L, R)$  and the set of orphans. The set of orphans, denoted  $\Omega(L, R)$ , is called the *fundamental domain* of the semigroup  $\langle L, R \rangle$ .

An *infinite path* in the forest  $\mathcal{F}(L, R)$  is a sequence  $(w_n)_{n=1}^\infty$  of complex numbers in  $\mathcal{D}_0$  such that  $w_{n+1} = L(w_n)$  or  $w_{n+1} = R(w_n)$  for all  $n \in \mathbf{N}$ . A *cuspid* of the semigroup  $\langle L, R \rangle$  is the limit of an infinite path in  $\mathcal{F}(L, R)$ . Thus,  $w^*$  is a cusp if  $w^* = \lim_{n \rightarrow \infty} w_n$ , where  $(w_n)_{n=1}^\infty$  is a path in  $\mathcal{D}_0$ . Of course, not every infinite path has a limit.

For every left-right pair  $(L, R)$  of matrices in  $SL_2(\mathbf{N}_0)$ , we have the following problems:

- (1) Compute the fundamental domain  $\Omega(L, R)$ .
- (2) Determine if the forest  $\mathcal{F}(L, R)$  contains infinite binary trees without roots, and describe them.
- (3) Determine the cusps of the semigroup  $\langle L, R \rangle$ .

## 2. EXAMPLE: TREES IN THE FOREST $\mathcal{F}(L_u, R_v)$

Let  $u$  and  $v$  be positive integers, and let  $(L_u, R_v)$  be the left-right pair of matrices defined by (2). The special case  $u = v = 1$  is a complex version of the Calkin-Wilf tree.

Let  $u > 0$ . For every positive integer  $n$ , we define the open half disk

$$\begin{aligned} \mathcal{D}_n &= \left\{ x + yi \in \mathcal{D}_0 : \left( x - \frac{1}{2nu} \right)^2 + y^2 < \left( \frac{1}{2nu} \right)^2 \right\} \\ &= \{ x + yi \in \mathcal{D}_0 : nu(x^2 + y^2) < x \}. \end{aligned}$$

We have the region

$$\mathcal{D}_0 \setminus \mathcal{D}_1 = \left\{ x + yi \in \mathcal{D}_0 : \frac{x}{u(x^2 + y^2)} \leq 1 \right\}$$

and, for  $n \geq 1$ , the half crescents

$$\mathcal{D}_n \setminus \mathcal{D}_{n+1} = \left\{ x + yi \in \mathcal{D}_0 : n < \frac{x}{u(x^2 + y^2)} \leq n + 1 \right\} \neq \emptyset.$$

The half disks and half crescents satisfy the relations

$$\mathcal{D}_0 \supset \mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots \supset \mathcal{D}_n \supset \mathcal{D}_{n+1} \supset \cdots,$$

$$\bigcap_{n=1}^{\infty} \mathcal{D}_n = \emptyset,$$

and, for  $0 \leq n < m$ ,

$$(\mathcal{D}_n \setminus \mathcal{D}_{n+1}) \cap \mathcal{D}_m = \emptyset.$$

If  $w$  is in the half disk  $\mathcal{D}_n$ , then

$$|w| \leq \left| w - \frac{1}{2nu} \right| + \frac{1}{2nu} < \frac{1}{nu}.$$

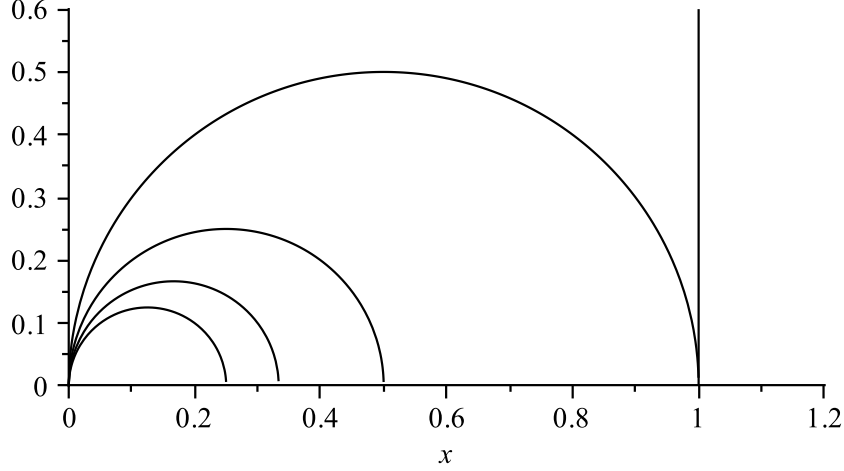


FIGURE 1. The half crescents for  $u = 1$  and  $n = 1, 2, 3, 4$  and the half plane  $\Re(z) \geq v = 1$ .

**Lemma 5.** *Let  $z \in \mathcal{D}_0$ , and let  $u$ ,  $v$ , and  $n$  be positive integers. Then  $\mathbf{R}_v^{-n}(z) \in \mathcal{D}_0$  if and only if  $\Re(z) > z + nv$ , and  $L_u^{-n}(z) \in \mathcal{D}_0$  if and only if  $z \in \mathcal{D}_n$ .*

*Proof.* Let  $z = x + yi \in \mathcal{D}_0$ . Because  $x > 0$  and  $y > 0$ , we have

$$R_v^{-n}(z) = z - nv = x - nv + yi \in \mathcal{D}_0$$

if and only if  $\Re(R_v^{-n}(z)) = x - nv > 0$ .

Let  $w = L_u^{-n}(z)$ . Note that both  $y$  and

$$|1 - nuz|^2 = (1 - nux)^2 + (nuy)^2$$

are positive real numbers. We have

$$\begin{aligned} w = L_u^{-n}(z) &= \frac{z}{1 - nuz} \\ &= \frac{z(1 - nu\bar{z})}{|1 - nuz|^2} = \frac{z - nu|z|^2}{|1 - nuz|^2} \\ &= \frac{x - nu(x^2 + y^2)}{|1 - nuz|^2} + \frac{yi}{|1 - nuz|^2} \in \mathbf{Q}(i) \end{aligned}$$

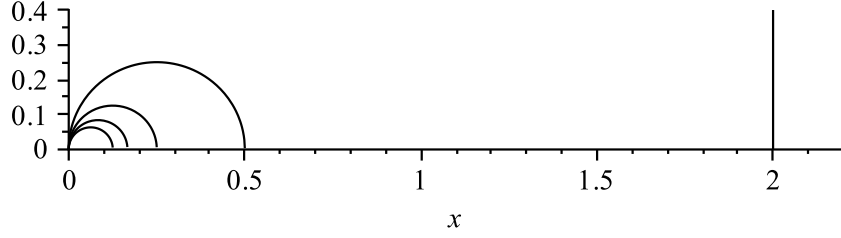


FIGURE 2. The half crescents for  $u = 2$  and  $n = 1, 2, 3, 4$  and the half plane  $\Re(z) \geq v = 2$ .

and  $\Im(w) = y/|1 - nu z|^2 > 0$ . It follows that  $w \in \mathcal{D}_0$  if and only if  $\Re(w) > 0$  if and only if

$$nu(x^2 + y^2) < x.$$

Completing the square, we see that  $w \in \mathcal{D}_0$  if and only if

$$\left(x - \frac{1}{2nu}\right)^2 + y^2 < \left(\frac{1}{2nu}\right)^2.$$

Thus,  $L_u^{-n}(z) \in \mathcal{D}_0$  if and only if  $z \in \mathcal{D}_n$ . This completes the proof.  $\square$

**Theorem 3.** *Let  $u$  be a positive integer. The linear fractional transformation*

$$L_u(z) = \frac{z}{uz + 1}$$

*maps  $\mathcal{D}_n \setminus \mathcal{D}_{n+1}$  onto  $\mathcal{D}_{n+1} \setminus \mathcal{D}_{n+2}$  for all integers  $n \geq 0$ .*

*Proof.* Let  $n \geq 0$ . If  $z \in \mathcal{D}_n \setminus \mathcal{D}_{n+1}$  and  $w = L_u(z)$ , then Lemma 5 implies that  $L_u^{-n-1}(w) = L_u^{-n}(z) \in \mathcal{D}_0$  and so  $w \in \mathcal{D}_{n+1}$ . If  $w \in \mathcal{D}_{n+2}$ , then  $L_u^{-n-1}(z) = L_u^{-n-2}(w) \in \mathcal{D}_0$  and so  $z \in \mathcal{D}_{n+1}$ , which is absurd. Therefore,  $L_u(z) \in \mathcal{D}_{n+1} \setminus \mathcal{D}_{n+2}$ .



Conversely, let  $w \in \mathcal{D}_{n+1} \setminus \mathcal{D}_{n+2}$ , and let  $z = L_u^{-1}(w)$ . Lemma 5 implies that  $L_u^{-n}(z) = L_u^{-n-1}(w) \in \mathcal{D}_0$  and so  $z \in \mathcal{D}_n$ . If  $z \in \mathcal{D}_{n+1}$ , then  $L_u^{-n-2}(w) = L_u^{-n-1}(z) \in \mathcal{D}_0$  and so  $w \in \mathcal{D}_{n+2}$ , which is absurd. Therefore,  $z \in \mathcal{D}_n \setminus \mathcal{D}_{n+1}$ , and  $L_u(z) = w$ . It follows that the function

$$L_u : \mathcal{D}_n \setminus \mathcal{D}_{n+1} \rightarrow \mathcal{D}_{n+1} \setminus \mathcal{D}_{n+2}$$

is onto. This completes the proof.  $\square$

**Theorem 4.** *For all positive integers  $u$  and  $v$ , the fundamental domain of the left-right pair  $(L_u, R_v)$  is*

$$\begin{aligned} \Omega(L_u, R_v) &= \{z \in \mathcal{D}_0 : z \notin \mathcal{D}_1 \text{ and } \Re(z) \leq v\} \\ &= \{x + yi \in \mathcal{D}_0 : u(x^2 + y^2) \geq x \text{ and } x \leq v\}. \end{aligned}$$

*Proof.* Let  $z = x + yi \in \mathcal{D}_0$ . Then  $R_v^{-1}(z) = (x - v) + yi \in \mathcal{D}_0$  if and only if  $x > v$ . Similarly,  $L_u^{-1}(z) \in \mathcal{D}_0$  if and only if  $z \in \mathcal{D}_1$ . Thus, the complex number  $z$  in  $\mathcal{D}_0$  has a parent if and only if either  $\Re(z) > v$  or  $z \in \mathcal{D}_0$ . Equivalently,  $z$  is an orphan if and only if  $z \in \mathcal{D}_0 \setminus \mathcal{D}_1$  and  $\Re(z) > v$ . This completes the proof.  $\square$

Bumby [3] and Thiel [17] have independently proved that every complex number in  $\mathcal{D}_0$  is descended from an orphan with respect to the left-right pair  $(L_u, R_v)$ .

**Theorem 5.** *For all positive integers  $v$  and  $u$ , the set of cusps of the semigroup  $\langle L_u, R_v \rangle$  is  $\{0, \infty\}$ .*

*Proof.* Let  $(w_n)_{n=1}^\infty$  be an infinite path in the forest  $\mathcal{F}(L_u, R_v)$ . If  $w_{n+1} = R_v(w_n)$  for all  $n \geq n_0$ , then

$$w_n = R_v^{n-n_0}(w_{n_0}) = \Re(w_0) + (n - n_0)v + \Im(w_0)i$$

and so  $\lim_{n \rightarrow \infty} w_n = \infty$ .

If  $w_{n+1} = L_u(w_n)$  for all  $n \geq n_0$ , then

$$w_n = L_u^{n-n_0}(w_{n_0}) = \frac{w_{n_0}}{(n - n_0)uw_{n_0} + 1}$$

and so  $\lim_{n \rightarrow \infty} w_n = 0$ .

For all  $w \in \mathcal{D}_0$ , we have  $L_u(w) \in \mathcal{D}_1$  and so

$$\Re(L_u(w)) < \frac{1}{u} \leq 1.$$

Similarly, for all  $w \in \mathcal{D}_0$ , we have

$$R_v(w) - w = v \geq 1.$$

If  $(w_n)_{n=1}^\infty$  is an infinite path in the forest  $\mathcal{F}(L_u, R_v)$  such that  $w_{n+1} = L_u(w_n)$  for infinitely many  $n$  and  $w_{n+1} = R_v(w_n)$  for infinitely many  $n$ , then  $w_{n+1} \in \mathcal{D}_1$  infinitely often and  $w_{n+1} - w_n = v \geq 1$  infinitely often. It follows that  $\lim_{n \rightarrow \infty} w_n$  does not exist. Therefore, the set of cusps of the semigroup  $\langle L_u, R_v \rangle$  is  $\{0, 1\}$ . This completes the proof.  $\square$

## 3. OPEN PROBLEMS

- (1) Classify the left-right pairs in  $SL_2(\mathbf{N}_0)$ .
- (2) Determine the left-right pairs  $(L, R)$  whose associated forests contain infinite binary trees without roots.
- (3) Let  $u, v \in \mathbf{N}$  with  $(u, v) \neq (1, 1)$ . Find an algorithm to determine if a matrix belongs to the semigroup generated by  $L_u$  and  $R_v$ .
- (4) Is there an efficient algorithm to determine if two numbers in  $\mathcal{D}_0$  are in the same tree?
- (5) Construct a class of freely generated subsemigroups of  $SL_2(\mathbf{N})$  of rank  $k \geq 3$ , and describe their associated forests of  $k$ -regular trees of positive complex numbers.

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